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Direct Numerical Computation of Periodic Orbits and Their Stability

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Introduction

THE computation of periodic orbits is very important for a variety of space systems [1]. The typical method used to compute periodic solutions is the Poincaré continuation method, which relies on the propagation of the nonlinear state equations. A predictor-corrector version of Poincaré's continuation method was presented by Lara and Peláez [2], which also generates the monodromy matrix necessary for computation of stability characteristics [3]. For complex or sensitive systems, these methods become difficult to apply because of possible instabilities inherent in the equations of motion and the difficulty in computing analytic derivatives. Furthermore, such techniques cannot be applied at all for computing controlled periodic orbits, such as those that may be required for maintaining spacecraft formations within acceptable constraints [4]. To deal with controlled periodic trajectories, optimal periodic control theory can be applied. Speyer [5] has summarized many of the advances in periodic optimal control theory as it relates to flight and cruise performance and also codeveloped a second variational theory for testing the optimality of periodic trajectories [6]. Periodic optimal control in this setting has been applied to obtain fuel optimal trajectories for aircraft [7] and hypersonic vehicles [8,9]. Typically, numerical shooting methods based on the necessary conditions for optimality are used to solve the two-point boundary value problem arising in such problems [10].

In this Note, the framework for generating periodic trajectories presented by Ross et al. [4] is extended to show how stability computations can be performed with little labor. The framework is based on direct discretization of the system equations using a Legendre pseudospectral method [11,12]. Using this method, it is demonstrated that the monodromy matrix for the periodic solution can be obtained naturally from the discretized Jacobian. Thus, it is shown how it is possible to obtain numerical solutions for both uncontrolled and optimally controlled periodic processes, and to assess the stability of the solutions without any formal derivations of the necessary conditions, calculation of costates, construction of transition matrices, or analytic derivations of the perturbed state equations. The key to being able to do this effectively is using the Legendre pseudospectral method as a direct transcription method. The Legendre pseudospectral method discretizes the continuous problem by expanding the states and controls using Lagrange interpolating polynomials. The Legendre–Gauss–Lobatto (LGL) points (roots of the derivative of an Nth degree Legendre polynomial) are used as collocation points, and the continuous optimal control problem is converted into a parameter optimization problem. In the case of a periodic process, the search for a periodic solution becomes one of solving a system of nonlinear equations. The key property of the approach is that the left- and right-hand sides (tangent bundle and vector field, respectively) of the system differential equations are approximated separately. This allows linear ordinary differential equations to be solved via a system of linear algebraic equations. This fact will be used to provide rapid solutions for the monodromy matrix of the periodic solution.

Numerical Method

The fundamental approach taken to generate periodic orbits is based on direct transcription [13]. Direct transcription methods are popularly used for trajectory optimization and optimal control because of the ease and speed with which they can be applied to complex systems. In fact, it is relatively straightforward to apply any direct transcription method to generate periodic orbits. Not only can natural periodic orbits be determined, but it is possible to determine optimally controlled periodic orbits. That is, orbits that are repeating, but which do not occur naturally and require active open-loop control to maintain. Practical examples of such orbits have been obtained for spacecraft formations [4,14] as well as tethered satellites in elliptical orbits [15]. Although, in principle, any discretization methodology can be used, certain discretizations lead to higher accuracy. For example, an Euler strategy could be employed, but this has very poor (linear) convergence rates. On the other hand, discretizations such as Runge–Kutta or Hermite–Simpson [13] do not allow the direct computation of stability characteristics. Here, it is shown that the use of a pseudospectral method generates both the periodic orbit and stability analysis in a natural way. For the purposes of brevity, the Legendre pseudospectral method is used, although the result can be extended to Chebyshev [16] or Jacobi [17] basis polynomials.

Consider the problem of finding a periodic orbit $\boldsymbol{x}(\boldsymbol{p},t) = \boldsymbol{x}(\boldsymbol{p},t+T)$, and possibly the corresponding control input $\boldsymbol{u}(\boldsymbol{p},t) = \boldsymbol{u}(\boldsymbol{p},t+T)$ and initial and final times t_0 and t_f ($T = t_f - t_0$) such that the cost function

$$J = \int_{t_0}^{t_f} \mathcal{L}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}, t_0, t_f, t]\,\mathrm{d}t \tag{1}$$

is minimized subject to the dynamical constraints

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}, t],\; t \in [t_0, t_f] \tag{2}$$

and periodicity constraints

$$\boldsymbol{x}(t_0) = \boldsymbol{x}(t_f), \qquad \boldsymbol{u}(t_0) = \boldsymbol{u}(t_f) \tag{3}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$ are the states, $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$ are the controls, $\boldsymbol{p} \in \mathbb{R}^{n_p}$ is a vector of parameters on which the periodic orbit depends, $\mathcal{L}$: $\mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the integrand of the selected cost function, and $t \in \mathbb{R}$ is the time. Although considered

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elsewhere [4,14,15] but not here, it is also possible to apply path constraints and general bounding box constraints to the formulation.

The problem is converted into a parameter optimization problem in the case where the cost function [Eq. (1)] and controls are nonzero, or a root-finding problem in the case where natural periodic orbits are sought. The Legendre pseudospectral method is applied for this purpose. The states and controls are expanded based on Lagrange interpolating polynomials

$$\boldsymbol{x}^N(t) \approx \sum_{j=0}^{N} \boldsymbol{x}_j \phi_j(t), \qquad \boldsymbol{u}^N(t) \approx \sum_{j=0}^{N} \boldsymbol{u}_j \phi_j(t) \tag{4}$$

The coefficients $\boldsymbol{x}_j = \boldsymbol{x}(t_j), \boldsymbol{u}_j = \boldsymbol{u}(t_j)$ in Eq. (4) are the values of the states and controls at the LGL points, which are the zeros of the derivative of the Nth order Legendre polynomial L_N defined on the interval $\tau \in [-1, 1]$. The Lagrange interpolating polynomials are defined by

$$\phi_j(\tau) = \frac{(\tau^2 - 1)\dot{L}_N(\tau)}{(\tau - \tau_j)N(N+1)L_N(\tau_j)}, \qquad j = 0, \ldots, N \tag{5}$$

The state derivatives are approximated by analytically differentiating Eq. (4) and evaluating the result at the LGL points with the result expressible in terms of the differentiation matrix $\boldsymbol{D}$, whose components are defined by

$$D_{k,j} := \begin{cases} \frac{L_N(\tau_k)}{L_N(\tau_j)} \frac{1}{(\tau_k - \tau_j)} & k \neq j \\ -\frac{N(N+1)}{4} & k = j = 0 \\ \frac{N(N+1)}{4} & k = j = N \\ 0 \quad \text{otherwise} & \end{cases} \tag{6}$$

The derivatives of the states at the collocation points are easily expressed by the following relationship,

$$\dot{\hat{\boldsymbol{x}}} \approx \frac{1}{\xi} \boldsymbol{D}\hat{\boldsymbol{x}} \tag{7}$$

where $\hat{\boldsymbol{x}} \triangleq [\boldsymbol{x}_0, \ldots, \boldsymbol{x}_N]$ is the discretized state vector across all nodes, and ξ is the transformation metric defined by the relationship between the computational domain τ and the physical time domain t

$$t = (t_f - t_0)\tau/2 + (t_0 + t_f)/2 \tag{8}$$

$$\xi \triangleq \frac{\mathrm{d}t}{\mathrm{d}\tau} = (t_f - t_0)/2 \tag{9}$$

Finally, the integral cost function is discretized via a Gauss–Lobatto quadrature rule so that

$$\int_{t_0}^{t_f} F(t)\,\mathrm{d}t = \xi \int_{-1}^{1} F[t(\tau)]\,\mathrm{d}\tau \approx \xi \sum_{j=0}^{N} w_j F(t_j) \tag{10}$$

where w_j are the Legendre–Gauss–Lobatto weights defined by

$$w_j = \frac{2}{N(N+1)} \frac{1}{[L_N(\tau_j)]^2}, \qquad k = 0, \ldots, N \tag{11}$$

For further details of the method, as well as its application to optimal control problems, please see [11,12]. Thus, the discretized constraint equations given by Eqs. (2) and (3) are expressed by

$$\frac{1}{\xi} \sum_{j=0}^{N} D_{k,j}\boldsymbol{x}_j - \boldsymbol{f}[\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{p}, t_k] = \boldsymbol{0}, \qquad k = 0, \ldots, N \tag{12}$$

$$\boldsymbol{x}_0 - \boldsymbol{x}_N = \boldsymbol{0}, \qquad \boldsymbol{u}_0 - \boldsymbol{u}_N = \boldsymbol{0} \tag{13}$$

In essence, because we are seeking smooth periodic orbits, the discretization converges at a spectral rate. The convergence of the method is understood from the point of view of a Sobolev space, denoted by $W^{m,p}(\Omega, \mathbb{R})$, which consists of all functions $f: \mathbb{R} \supseteq \Omega \to \mathbb{R}$ whose jth derivative is in L^p for all $0 \leq j \leq m$, and L^p represents the standard L^p-norm. The method converges according to the following theorem proved in [18].

Theorem: Let $\boldsymbol{x}^*(t) \in W^{m_x,\infty}([t_0, t_f], \mathbb{R}^{n_x})$ be the optimal periodic state trajectory, and $\boldsymbol{u}^*(t) \in W^{m_u,\infty}([t_0, t_f], \mathbb{R}^{n_u})$ be the corresponding optimal periodic control. Under the proper set of technical conditions, the following convergence holds,

$$\begin{aligned} \|\boldsymbol{x}^*(t) - \boldsymbol{x}^N(t)\|_{L^\infty} &= \mathcal{O}(N^{-m_x}), \\ \|\boldsymbol{u}^*(t) - \boldsymbol{u}^N(t)\|_{L^\infty} &= \mathcal{O}(N^{-m_u}) \end{aligned} \tag{14}$$

where m_x and m_u are positive integers that depend on the smoothness of the solution.

Hence, the method convergences at an exponential (spectral) rate. The convergence of the pseudospectral (PS) method is far superior to any other discretization that can be applied to the problem, provided the solution is smooth. It is necessary to ensure an accurate solution so that reliable stability calculations can be made.

Stability Computations

An effective method of computing the stability of the periodic solution is to use Floquet's theory on the variational equations derived by perturbing the periodic orbit [3],

$$\delta\dot{\boldsymbol{x}}(t) = [A(t)]\delta\boldsymbol{x}(t) \tag{15}$$

where $A_{k,j} \triangleq \partial f_k/\partial x_j$ are continuous periodic functions with period T and are the elements of the matrix $[\boldsymbol{A}(t)]$. Equations (15) are linear and homogeneous. Floquet's theory gives the stability properties of the periodic orbit as a function of the eigenvalues λ_i, $i = 1, \ldots, n_x$ of the monodromy matrix $[M]$. Let $[\varphi(t)]$ be a fundamental matrix of the Eq. (15), then any solution of Eq. (15) can be written as

$$\delta\boldsymbol{x}(t) = [\varphi(t)]\delta\boldsymbol{x}_0 \tag{16}$$

Consequently, the monodromy matrix is determined by letting $[\varphi(t_0)] = \boldsymbol{I}_{n_x \times n_x}$ to obtain $[M] = [\varphi(T)]$. Practically, the monodromy matrix is often computed by carrying out n_x integrations of Eq. (15) with initial conditions corresponding to the jth column of $\boldsymbol{I}_{n_x \times n_x}$, $j = 1, \ldots, n_x$. The algorithm presented by Lara and Peláez [2] uses the variational equations to compute the corrections to the periodic orbit and hence the monodromy matrix is automatically available to compute stability properties. If all the eigenvalues of the monodromy matrix satisfy $|\lambda_i| < 1$, $i = 1, \ldots, n_x$ then the periodic orbit is asymptotically stable. However, if any one of the eigenvalues has $|\lambda_i| > 1$ then the periodic orbit is unstable. These properties of periodic trajectories have been used to optimize the open-loop stability of walking robots [19]. It will now be shown how the solution obtained via the Legendre pseudospectral method can be related to monodromy matrix.

Consider the discretization of Eq. (15) via the Legendre pseudospectral method,

$$\begin{aligned} &\frac{1}{\xi} \sum_{j=0}^{N} D_{k,j}\delta\boldsymbol{x}_j - [A(t_k)]\delta\boldsymbol{x}_k = \boldsymbol{0}, \qquad k = 0, \ldots, N \\ &\delta\boldsymbol{x}_0 = \delta\boldsymbol{x}(0) \end{aligned} \tag{17}$$

which can be written in block matrix form as

$$
[F]z = \begin{bmatrix} \frac{1}{\xi}\boldsymbol{D} - \boldsymbol{A}_{1,1} & -\boldsymbol{A}_{1,2} & \cdots & -\boldsymbol{A}_{1,n_x} \\ -\boldsymbol{A}_{2,1} & \frac{1}{\xi}\boldsymbol{D} - \boldsymbol{A}_{2,2} & \cdots & -\boldsymbol{A}_{2,n_x} \\ \vdots & \vdots & \vdots & \vdots \\ [1,0,\ldots 0] & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & [1,0,\ldots 0] & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & [1,0,\ldots 0] \end{bmatrix} \begin{bmatrix} (\delta\hat{x})_1 \\ (\delta\hat{x})_2 \\ \vdots \\ (\delta\hat{x})_{n_x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta\boldsymbol{x}(0) \end{bmatrix} \tag{18}
$$

where $\boldsymbol{A}_{k,j} = \text{diag}[A_{k,j}(t_0), \ldots, A_{k,j}(t_N)]$ is a diagonal matrix that contains the entries of the variational equations at all LGL nodes evaluated on the periodic orbit, and $(\hat{x})_j = [x_1, x_2, \ldots, x_N]_j$, $j = 1, \ldots, n_x$, i.e., the values of the jth state at the LGL nodes. The matrix on the left-hand side of Eq. (18), $[F]$, is nothing more than the Jacobian of the discretized constraints given in Eqs. (12) and (13) and with respect to the states [note a minor modification is necessary to set the derivatives of Eq. (13) with respect to $\boldsymbol{x}_N$ to zero]. The monodromy matrix can be formed in one of two ways: either by solving Eq. (18) n_x times with the appropriate initial conditions to obtain the values of the states at the final time $\delta\boldsymbol{x}_N$, or by rearranging Eqs. (18) and eliminating the first row in each block of $[F]$ containing the differentiation matrix $\boldsymbol{D}$, which renders the resulting $[F^*]$ matrix square with dimension $n_x(N+1)$. Assuming that the inverse of $[F^*]$ exists, let it be written as

$$
[F^*]^{-1} = G \tag{19}
$$

Note that the existence of the inverse of $[F^*]$ cannot be guaranteed in general. In some situations, $[F^*]$ can become singular or nearly singular and it is necessary to use more accurate methods such as LU-decomposition to solve the system of equations. In other cases, it may be necessary to relax the equalities in Eq. (18) to inequalities with a small tolerance. For more discussion on the existence of solutions via pseudospectral methods and the relaxation of equality constraints, see [20]. The monodromy matrix can be obtained by extracting the elements of G as follows:

$$
M = \begin{bmatrix} G_{(N+1),1} & G_{(N+1),(N+1)+1} & \cdots & G_{(N+1),(n_x-1)(N+1)+1} \\ G_{2(N+1),1} & G_{2(N+1),(N+1)+1} & \cdots & G_{2(N+1),(n_x-1)(N+1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ G_{n_x(N+1),1} & G_{n_x(N+1),(N+1)+1} & \cdots & G_{n_x(N+1),(n_x-1)(N+1)+1} \end{bmatrix} \tag{20}
$$

Equation (20) relates the terminal values of the perturbed system in Eq. (15) to the initial conditions. Thus, the transformation is achieved without any explicit integration or additional analytic computations. The monodromy matrix can be formed directly from the Jacobian of the discretized nonlinear equations. The practical benefit of this is that virtually all nonlinear programming implementations compute the Jacobian. In this work, the software SNOPT [21] is used as the nonlinear programming solver. Another major benefit of this approach is that the Jacobian derivatives can be computed via finite differences. Thus, for highly complex systems where analytic derivatives are difficult to compute, the method can be easily applied. Furthermore, another advantage of the method is that no propagation of equations of motion is necessary and hence sensitivity problems are not encountered. In addition, no continuation is necessary to obtain solutions due to global convergence of many NLPs including SNOPT [21]. However, if tracing the path of a particular family of solutions is desirable, continuation can still be applied by using a previous solution as initial guess and applying a perturbation to the vector of parameters $\boldsymbol{p}$. In this case, much larger continuation steps can be taken than conventional algorithms.

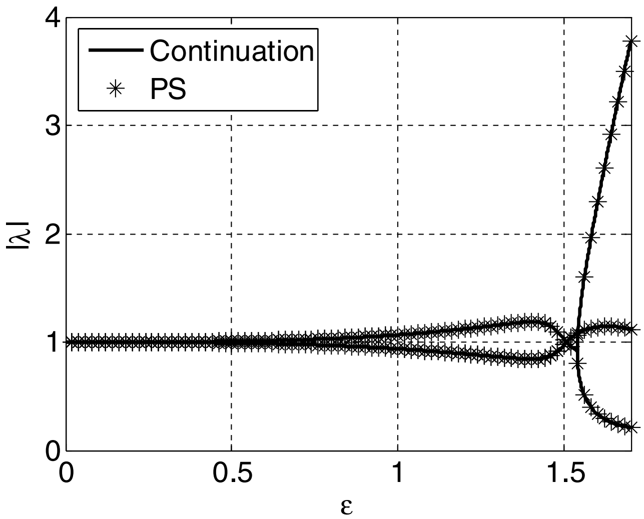

Fig. 1 Modulus of eigenvalues of monodromy matrix for electrodynamic tether example for $i = 25$ deg.

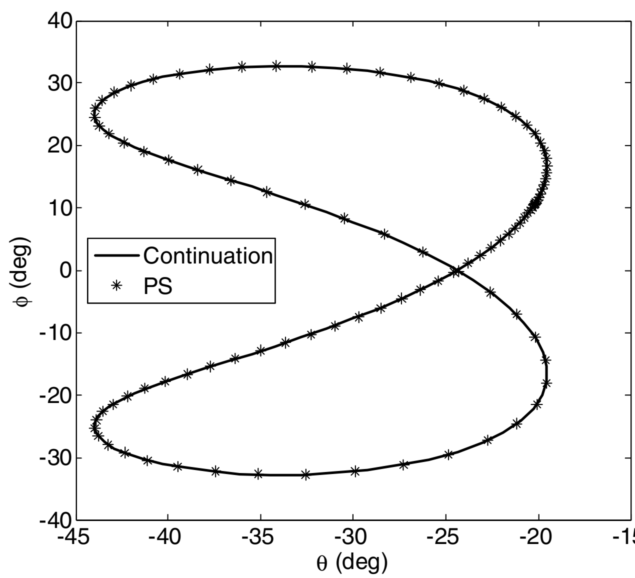

Fig. 2 Periodic solution for electrodynamic tether with $\varepsilon = 1.7$ and $i = 25$ deg.

Numerical Example

To demonstrate the power of the method, an example of the computation of periodic trajectories for an electrodynamic tether system in a circular inclined orbit is considered, taken from [22]. The equations of motion representing the librational motion of the tether are given by

$$
\theta'' = 2(\theta' + 1)\phi' \tan\phi - 3\sin\theta\cos\theta - \varepsilon[\sin i \tan\phi(2\sin\nu\cos\theta - \cos\nu\sin\theta) + \cos i] \tag{21}
$$

$$
\phi'' = -[(\theta' + 1)^2 + 3\cos^2\theta]\sin\phi\cos\phi + \varepsilon\sin i(2\sin\nu\sin\theta + \cos\nu\cos\theta) \tag{22}
$$

where ε is a nondimensional parameter that depends on the electric current and mass distribution in the tether system, θ is the in-plane libration angle, ϕ is the out-of-plane libration angle, ν is the nondimensional time (true anomaly measured from the line of nodes), i is the orbit inclination, and $()'$ is the time derivative with respect to true anomaly. The modulus of the eigenvalues of the monodromy matrix are shown in Fig. 1 for the case $i = 25$ deg, which compares solutions via a predictor-corrector algorithm with the PS method with $N = 100$. An example periodic solution is shown in Fig. 2 for $\varepsilon = 1.7$ and $i = 25$ deg. It is evident that the PS method captures the periodic solution excellently. The results in Fig. 1 show that the method has accurately determined all of the stability properties of the solutions. The predictor-corrector approach is capable of dealing with the case where ε is constant, as shown in [22]. However, if the more practical case of a forced current variation is used, such as $\varepsilon = \varepsilon_0 \sin \nu$ (applicable for changing the orbit eccentricity of the tether system), then the predictor-corrector approach fails after one step due to sensitivity issues, whereas the present method converges quite readily. However, for the sake of brevity, results from these additional examples are not shown here.

Conclusions

A numerical approach for directly computing periodic orbits and their corresponding stability via pseudospectral methods has been presented. The method is able to generate periodic orbits without continuation, and is applicable to natural and forced periodic orbits. The approach relates the Jacobian of the discretized equations used in the determination of the periodic orbits to the monodromy matrix, which yields the stability information. This makes stability computations very efficient and effective for complex systems for which analytic derivatives may be difficult to construct. Furthermore, no propagation of equations of motion is necessary, thus alleviating difficulties associated with sensitivity issues.

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